

Borromean Braid Crossings from the Triaxial Grid

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Abstract

The Borromean rings consist of three rings, made from two initially unconnected rings and one threaded through them, joining all three into a link. Here, we interweave braids covering the lines of the triaxial grid such that removing one set of parallel braids completely, the other two fully separate, relating our findings to the Borromean rings. Aside from the construction, we present a collection of examples.

Borromean Rings

The Borromean rings are a prominent example of a link made of three unknots with the property that, upon deleting an arbitrary one, the link falls apart and two unknots remain. Known as being part of the coat of arms of the Borromean family, the motif itself appears in various cultural surroundings, with an example in Figure 1(a). In the coat of arms of the Borromean family, the rings symbolize the strong bond of three aristocratic families in Italy, while in Christian symbolism, the three linked rings represent the Trinity [4].

Considering the three-colored representation of the Borromean rings, Figure 1(b), we see the black ring crossing over the one colored in sand and the one colored in sand crossing over the blue one and the ring colored in blue crossing over the black one, which we denote by $\text{black} > \text{sand} > \text{blue} > \text{black}$. It is known from knot theory, that each link can be drawn into the plane such that at most two knots cross in a single point. The knot segments connecting such crossings enclose (possibly) unbounded areas, which we call *regions*, and the set of these segments is called the *border* of the region. In the representation of the Borromean rings, we find seven bounded triangular regions and a single unbounded one, each surrounded by three circular arcs. This representation induces a relation of the colors on the segments: In all regions, in cyclic order (either clockwise or counter-clockwise), black appears first, then sand, followed by blue. This order is fixed by the representation of the link in the plane and is reversed when flipping the link over. This order will later allow us to characterize the links that separate when removing an arbitrary color, for instance, Figure 1(e).

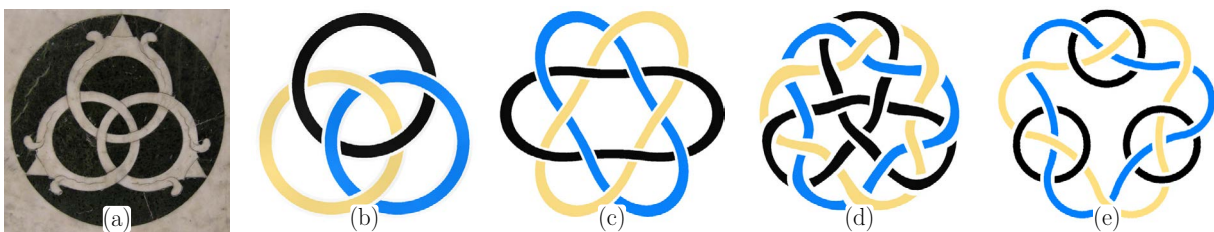


Figure 1: (a) Borromean rings as a decorative ornament (*opus sectile*, 15th century, Florence, Italy) [8]; (b) three-colored Borromean rings; (c) Brunnian link; (d) decomposable link (trefoil and two unknots); (e) decomposable link of several components, equally colored.

The Borromean rings belong to the set of the so-called *Brunnian links* [3], which consist of an arbitrary, finite number of unknots that are linked in such a way that removing an arbitrary one leads to all other unknots becoming unlinked. The Borromean rings are—with respect to the number of unknots used as well as to the number of crossings in a planar representation—the minimal example for a Brunnian link.

The properties recalled above have equally fascinated scientists and artists. While scientists work on the classification and creation of new types of Brunnian links [2], artists focus on slightly different aspects. They have used Brunnian links to make clothing [1] or decorate tiles [7]. Several periodic Brunnian links have been constructed based on the hexagonal tiling of the plane [6].

Borromean Crossings

In previous work, we presented patterns of interlaced strands made by replacing single lines in a weaving by braids [5]. We call such a structure a *braiding*. We focused on possibilities to cross two braids wherever two lines in the weaving crossed before they were replaced. We presented various ways to maintain the braiding structure within the crossings. Roughly speaking, we asked for each single strand to go alternately above and below other strands it meets. From now on, we refer to this property by calling it the *alternating property*.

While in the weavings used as bases, at most two lines meet at a point, this is not the case in the triaxial grid. Here, at each crossing, three lines meet. In the following, we present several possibilities to interlace three braids at a common point and—since we have three strands—we also aim at incorporating the properties of the Borromean rings observed above. That is, in addition to the alternating property, we would like the border of each region to consist of colored arcs obeying the cyclic relation $\text{black} > \text{sand} > \text{blue} > \text{black}$. We generalize the latter property to allow consecutive border segments to go through several cycles, as done twice in Figure 1(c), and to appear in the same color, as shown in the link made from a trefoil and two unknots in Figure 1(d). This property is called *cyclic property*. Above, we recalled that after deleting any of the three Borromean rings, two disconnected unknots remain. In the construction presented in this paper, we will achieve a slightly different result. Removing all strands in a single color, two separate sets of interwoven strands remain, one in each of the colors not removed, independent of the color we pick. We say that a link fulfilling both, the alternating and the cyclic property, obeys the *Borromean property*. The kagome or trihexagonal weave shown in Figure 2(c) exemplifies this.

For our construction, we start with the triaxial grid and color each of the three sets of parallel lines occurring in a single color, Figure 2(b). As done in the representation of the Borromean rings, we choose blue, black, and sand. Further, each line is replaced by a braid made of three strands, Figure 2(d), where each strand fulfills the alternating property internally, Figure 2(e). At those points, where three braids cross, we included a small section, where the strands run parallel to each other, as in Figure 2(d). The red circles indicate these crossing areas.

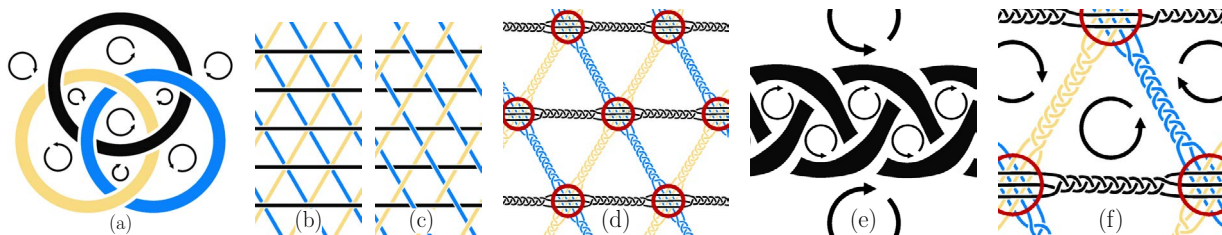


Figure 2: Set up construction of Borromean crossings. (a) Borromean rings with oriented regions; (b) triaxial grid; (c) kagome grid, in (b) and (c) parallel lines are colored individually; (d) lines replaced by braids; (e) oriented regions in braid section; (f) partially oriented regions, crossings to fix marked with red circles.

In the following section, we explore how crossings of three braids can be created to satisfy the alternating property and investigate whether they obey the Borrormean property as well. In case they do not, we introduce *fixing operations* to make the crossings Borrormean. In the end, our findings can be realized by placing the braids colored in blue and sand on top of each other and using the black strands alone to interweave them. During the development process, we also make use of strands not colored in black to create crossings of strands in blue and sand. Thus, we let the strands follow the course of the line they replace.

Since we assume the linked braids to fulfill the alternating property, the crossings of the strands induce an orientation in each region as depicted in Figure 2(a). Regions sharing a border section are oriented differently. Based on the internal structure of a braid made of three strands, Figure 2(e), the regions on both sides of the braid are also oriented differently. Therefore, the braids replacing the lines from the initial grid have to be placed such that the orientations of the former triangular regions fit together as illustrated in Figure 2(f).

Crossings Braids

We now want to create a braiding based on the triaxial grid that satisfies the Borrormean property as defined above. A starting point is to replace all lines in the triaxial grid from Figure 2(b) with braids. Then, a priori, it is not clear how to interweave the individual strands of the three differently colored braids that are meeting at a crossing of the triaxial grid. Especially, it is not clear how to do so such that the Borrormean property is maintained for the result. Hence, we first do not take any practical steps into creating such a construction, but replace the lines by braids, bring them together in a way that satisfies the alternating property and see what happens. For any crossing that is then not satisfying the cyclic property, we will fix it locally. Thereby, we will obtain a blueprint of what the final configuration will look like. Armed with such a blueprint, we can then tackle the actual physical realization.

In the following, we will focus on a crossing of the triaxial grid. We replace the blue line by a blue braid. As the sand line is on top of the blue one, we replace the sand line with a sand braid on top of the blue braid. Now we are left with replacing the black line with a black braid. If we just place the black braid over the sand one, it will violate the alternating property, just like the sand braid violates the alternating property with the blue braid. Thus, we have to weave the black braid in, going above and below the blue and sand strands, and have sand and blue also alternate. This can be done in an alternating fashion (Figure 3(a)–(d) show four different ways of doing so). However, not all resulting crossings satisfy the cyclic property. We call such a crossing a *defect* and highlight them with a red circle as done in the third row of Figure 3.

Comparing the two leftmost images in the third row of Figure 3 with the other two, we observe that the defects occurring there are part of borders containing all three colors, which is not the case in the other two images. Defects belonging to a border made of all three colors are called *integrated*, while the others are called *isolated*. In the next section, we introduce minimally invasive procedures to fix any defects.

Before we turn to these fixes, note that we presented four different ways, (a)–(d), of how to interweave the black braid with the other two in the second row of Figure 3. Yet, there are eight different ways of how to weave the black braid through the other two: Interweaving the blue and sand braid creates a diamond of blue and sand crossings, with one, two, three, two, and one crossings horizontally ordered above each other. We can choose to have the black braid below these, or shifting it up, such that a horizontal line of blue and sand crossings is between two black strands. As there are five horizontal lines of blue and sand crossings and three black strands, there are eight possibilities, which are shown in the first row of Figure 3. They include the four cases a–d, which correspond to the missing four by reversing the direction of the braid. This flips all crossings from being defect to defect-free and vice versa, compare for instance (a) and (a’).

We started this section with the question on how to include the Borrormean property in combined weavings and braidings, similar to prior work [5]. Now, we can come back to this motivation and make some observations. Focusing on the second row of Figure 3, each of the four images can be partitioned into areas

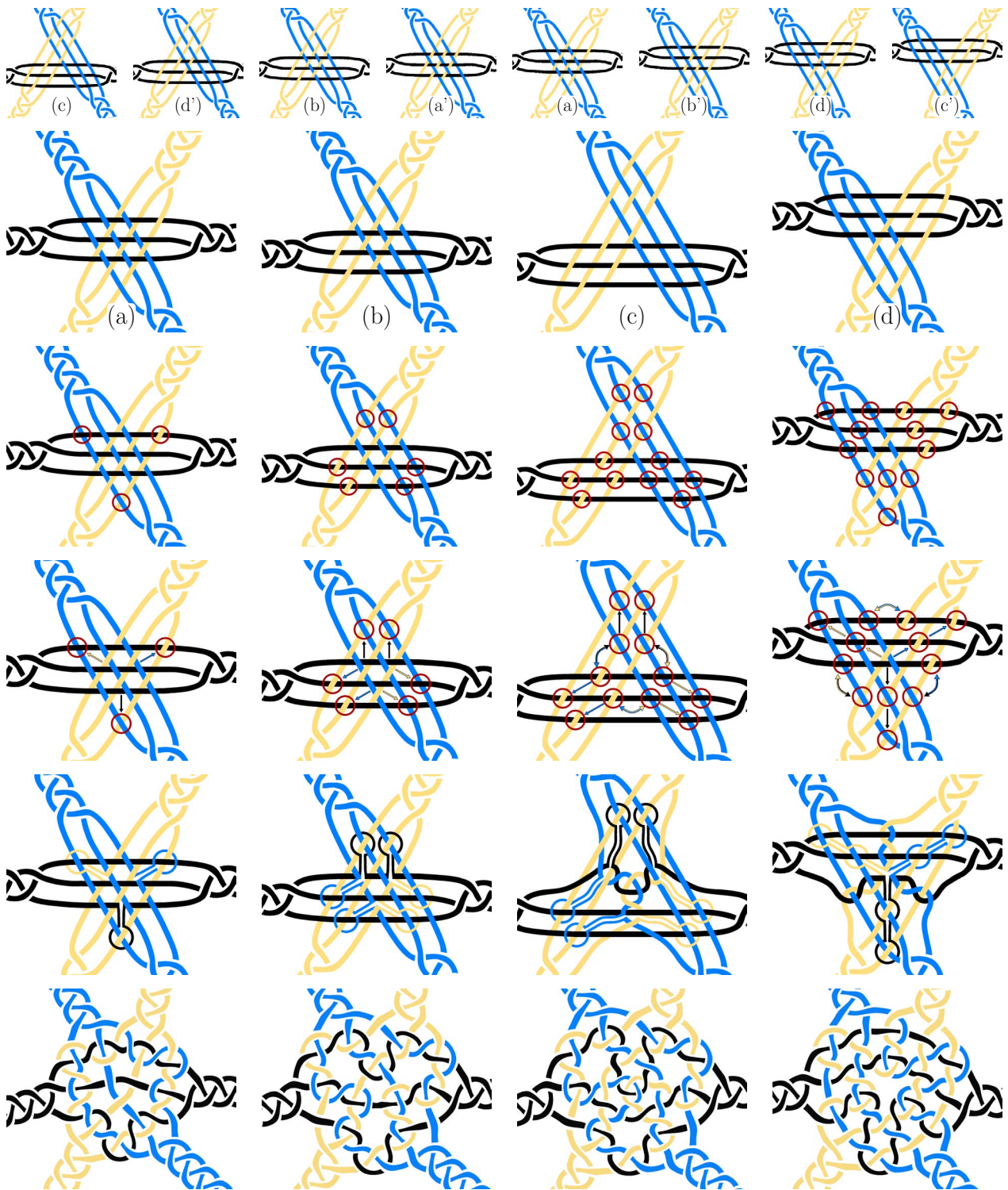


Figure 3: Creating Borromean crossings. Per row, top to bottom: 1st: eight possible starting configurations; 2nd: four of the eight chosen to start; 3rd: defects indicated by red circles; 4th: fixing operations indicated by arrows; 5th: diverted strands; 6th: smoothed final results.

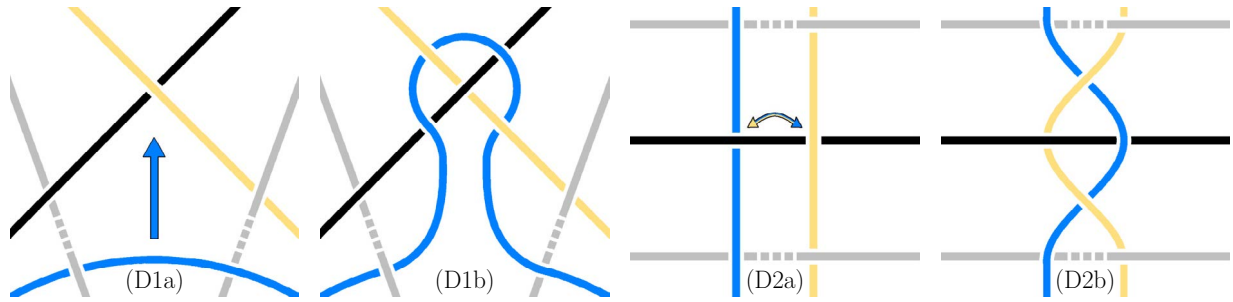


Figure 4: Fixing integrated defects: general procedure D1 and special case D2.

of three different types. These are kagome weave (in the center, with all three colors meeting), braids (on the outside, with a single color coming in), and plain weave (where two, not three braids meet). The kagome part does already satisfy the Borromean property and so do—by definition—the braided parts. The two-colored plain weave areas can never satisfy the cyclic property, though, so we make them at least alternating and will deal with their defects in the next section.

Fixing Defects

Note that there is only one type of integrated defect that we have to fix, which is a crossing that violates the cyclic property in a three-colored border. We will present a general procedure, labeled D1, to fix this defect. However, we will see that procedure D1 introduces four new crossings. In the special case of two neighboring integrated defects, we can therefore employ a slightly different procedure, labeled D2, that introduces only two new crossings while fixing both defects.

First, we introduce operation D1, applicable to integrated defects, where by construction all crossings satisfy the alternating property. Thus, at a defect, a crossing of two strands violates the cyclic property. Hence, there are two strands obeying the alternating property, but with the wrong color on top as shown in Figure 4(D1a). To fix this, we exchange the strands causing the defect. While we now moved the desired color on top, we violated the alternating property. So, pick a strand in the color not part of the defect, which appears in the same border as the defect, since we are working on fixing an integrated defect (color indicated in Figure 4(D1a) by an arrow). We change the third strand so that it alternates above and below the strands of the defect, creating four further crossings fitting around the one made of the exchanged strands as illustrated in Figure 4(D1b). This resolves the defect.

Introducing four new crossings to fix a single integrated defect might not always be desirable. In the special case that we have two neighboring crossings that both exhibit an integrated defect, Figure 4(D2a), we can employ a different procedure, which we will describe here. To start, we form an ‘eye’ with those strands causing two defects next to each other with the same strand, indicated by an arrow in Figure 4(D2a). Afterwards, we thread this strand through the eye such that all four crossings fulfill both properties, Figure 4(D2b). Thereby, we end up with replacing two defects by four crossings obeying the properties formulated in the beginning.

In both cases, D1 and D2, the lines colored in gray indicate a border segment with any number (including zero) of other strands in an arbitrary color. It does not matter whether the crossings with the gray strands are correct or not. In any case, they remain unchanged from the operation explained before. If there are further defects present (they might change from isolated to integrated), they have to be fixed in an additional step.

After applying procedure D1 or D2, a fixed crossing remains fixed after applying this procedure to other defects, since it is a local operation, changing solely one crossing present in the braiding. By construction, the crossings introduced by the fixing operation are not defects, which can be observed by focusing on the

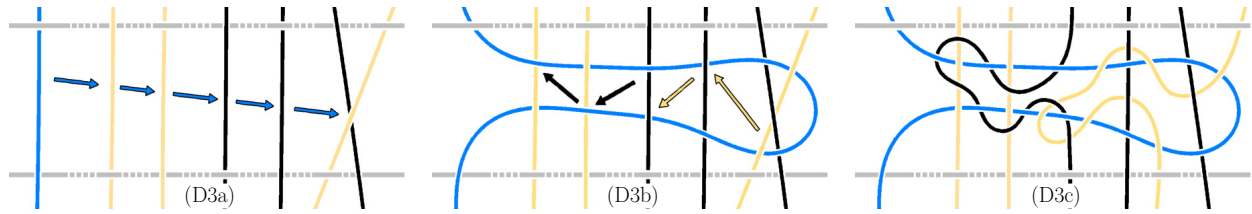


Figure 5: Fixing isolated defects (D3), illustrated by an example.

regions incident to the crossing. We do not create further defects, since the alternating property of all strands is untouched. Hence, applying procedure D1 or D2 strictly reduces the number of defects, which implies that repeated application of D1 and D2 fully removes all integrated defects.

Now, we turn to fixing isolated defects, which belong to borders that are made of exactly two colors. Borders of only one color are, as long as all their crossings are alternating, defect-free by the definition of the cyclic property. As our construction ensures alternating crossings, only borders of exactly two colors are left to be fixed. We will first go through an example, illustrated in Figure 5, showcasing the general procedure. In our example, the defect is located at the crossing of a black and a sand strand. As it is an isolated defect, there is no blue strand immediately available. We therefore find the combinatorially closest blue strand, which can be several black and sand strands away. In our example, there is a blue strand that is separated from the defect crossing by two sand and two black strands, Figure 5(D3a). As a first step in fixing the defect, we weave the blue thread through the separating strands to have it available to fix the defect as shown in procedure D1. This creates several new crossings and ensures that the alternating property remains intact for each of these, but the cyclic property is violated for these new crossings, Figure 5(D3b).

We can now iteratively work our way back from the crossing that we set out to fix in the first place to where we got the blue strand. The first defect we encounter is a crossing with black on top and blue below. From the now fixed crossing, we can fix this integrated defect by procedure D1, dragging the sand strand to fix it, as indicated by a sand-colored arrow in Figure 5(D3b). Continuing towards the original position of the blue strand, we encounter another black > blue defect. We fix it in the same manner with the now available sand strand, indicated by the second sand arrow in Figure 5(D3b). Similarly, we can fix the two remaining defects, as indicated by the black arrows. The final, defect-free situation is depicted in Figure 5(D3c).

This exemplifies the general procedure for fixing isolated defects: Bringing in a strand of the missing color creates a set of (probably isolated) defects. These become iteratively integrated as, starting from the fixed, formerly isolated, defect, the closest defect is integrated. Fixing it makes the second closest integrated and so on until all newly created defects are fixed. As for integrated defects, this operation introduces new crossings, but after finishing procedure D3, the isolated defect is gone and all new crossings are defect-free. Hence, a repeated application of D3 can fully remove all isolated effects from a construction.

With this knowledge, we return to Figure 3. From top to bottom, each of the four main columns shows: One of the eight initial configurations arising from a crossing of the triaxial grid; defect crossings highlighted with red circles; necessary fixing procedures to remove the defects; a minimal, rubber-band rendering of the fixed configuration; a smoothed rendering of the final configuration. This last row of Figure 3 can then finally serve as a blueprint for the creation of a physical realization of our construction.

In fact, each of the eight possible starting configurations provides a different blueprint that can be applied to a crossing of the triaxial grid. We can choose a different blueprint for each crossing as by parallel translation of the crossings, braid sections fit to strands of the same color entering or leaving a neighbored crossing—a property that sets this construction aside from prior work, where the braids flipped, depending on the crossings [5]. The choice of blueprints opens up some creative choices, but we can even alter the blueprints themselves.

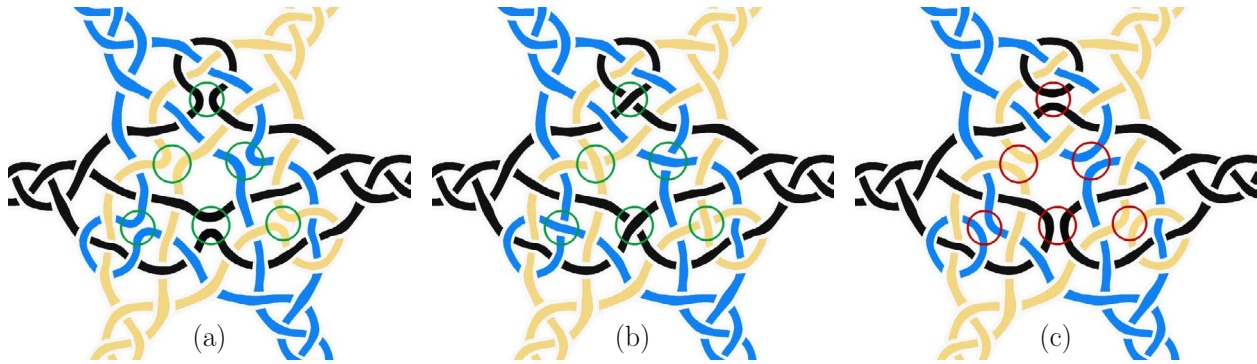


Figure 6: *Creating or removing crossings within one color.*

Artistic Freedom: Modifying the Blueprint and Gallery of Physical Realizations

The current set of eight different blueprints can be used to achieve some physical realizations. However, if we consider the entire set of Borromean weavings from braids, then there are many more. In this section, we present two operations—adding same-color crossings and adding braid segments—that preserve the Borromean property whether increasing or decreasing the combinatorial complexity.

So far, we have not considered crossing strands of the same color. However, we can easily introduce these, while keeping the Borromean property. Consider Figure 6(a) where green circles highlight pairs of same-colored border elements within a region. Each of these parallel border segments can be replaced by a crossing in the same color, Figure 6(b). In principle, the reverse is also true: Any crossing in a color can be replaced by two parallel segments in said color. However, one has to be careful not to create disconnected strands or have the braid loop back into itself, Figure 6(c).

Similar to crossings, we can insert entire braid segments. This is illustrated in Figure 7. First, we cut along one or more lines that each cross all three colors, Figure 7(a). Then a braid segment is inserted in each cut (Figure 7(b), the dashed lines indicate how to glue braid segments together). This provides another Borromean braiding, with additional braid segments, Figure 7(c).

While there are surely more operations that can be used to alter the blueprints, we present these two here to increase artistic freedom. Also, these serve to show that there is no complete classification of Borromean braidings as these can be arbitrarily extended. Based on these two operations, we present eight different physical realizations in Figure 8. In these, we did focus on the rotational symmetries present in the underlying triaxial grid and on maintaining these by choosing respective strands, where possible.

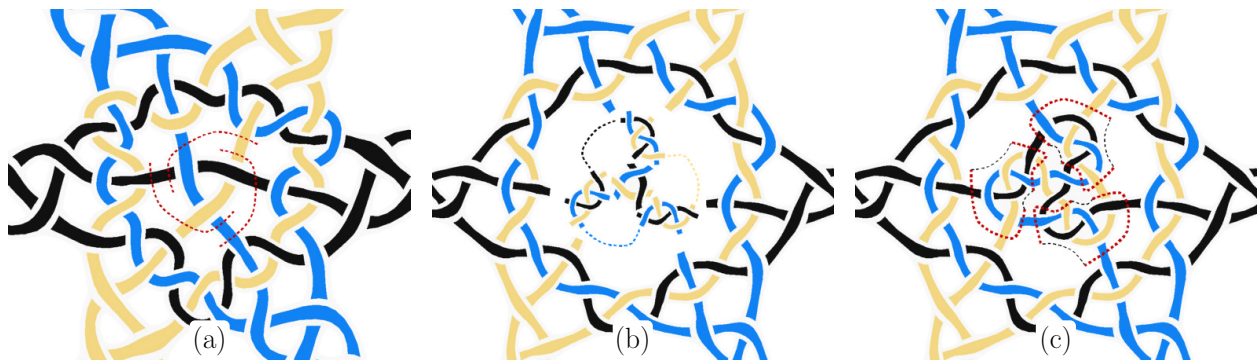


Figure 7: *Inserting braid segments.*

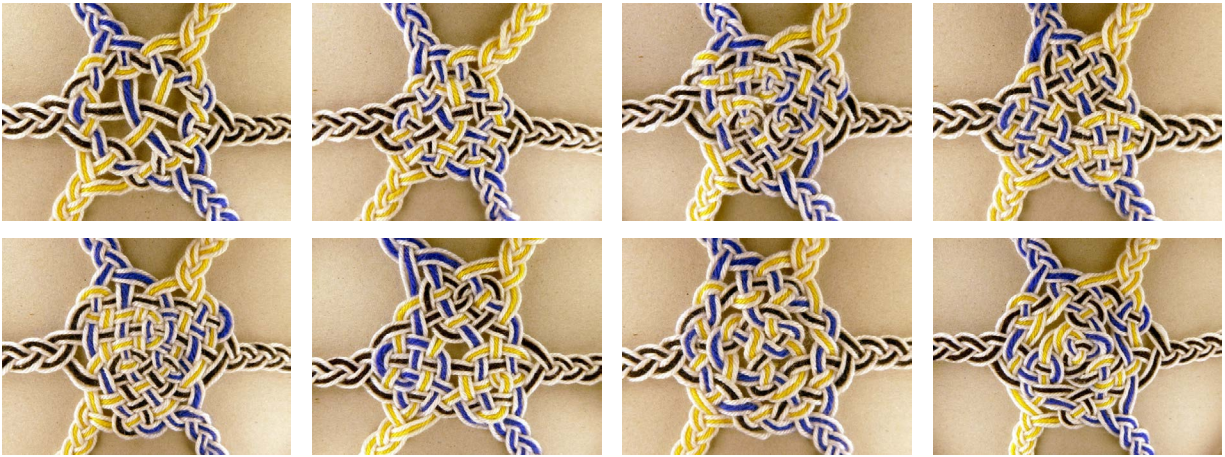


Figure 8: Physical realizations (blue, black, sand, and white cotton yarn): minimal number of crossings; continue braids in the crossing area; three crossings per color; heart per color; four crossings per color; pretzel shape per color; braid pieces inserted; center with three linked loops.

Summary and Conclusions

In this article, we replaced the lines in the regular triaxial grid by braids made of three strands and investigated ways to let them cross each other at the grid vertices. Inspired by the property of the Borromean rings, that deleting one of the rings lets the other two fall apart, we developed minimally invasive operations, fixing crossings violating the cyclic ordering of colors. In addition to creating these Borromean crossings, we also presented a collection of examples, made digitally and by hand.

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